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Induction Heating of Flat Objects

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Contents: It is a usual procedure in industry for various machine elements like metal plates of different thickness to be heated inductively with a frequency within the range 50 Hz to 3 MHz. The paper discusses the problem of finding the best method in this respect with the optimum heating parameters.

Three different induction heating systems of plates are analysed: unilateral and bilateral heating with compatible and inverse currents. The discussion, calculation and conclusions given in the paper can be also applied in reference to plates of an irregular shape.

Übersicht: Es ist in der Industrie üblich verschiedene Maschinenelemente wie Metallplatten von unterschiedlicher Dicke mit Frequenzen zwischen 50 Hz und 3 MHz induktiv zu erwärmen. Für diese Zwecke wird in diesem Aufsatz das Optimierungsproblem von Erwärmungsparametern betrachtet. Es werden drei verschiedene Induktionserwärmungssysteme von Platten analysiert: ein- und zweiseitige Erwärmung durch gleich- und entgegengerichtete Ströme. Die angegebenen Betrachtungen, Berechnungen und Ergebnisse können auch auf die Induktionserwärmung von zylindrischen Platten angewendet werden.

1. Introduction

Inductive heating very often means heating flat objects, e.g. metal plates of different thickness. These are usually elements which are heated with a frequency within the range 50 Hz to 3 MHz, either thoroughly with respect to the plastic working or to the heat treatment of the metal element, or are heated on the surface only for the purpose of hardening.

Flat metal objects as well as those of a cylindrical shape which are also frequently used are the typical machine elements used in induction heating. All the other possible shapes of objects can be nearly always reduced in theoretical and technical consideration to a flat or a cylindrical shape.

The elements in question are mostly objects of mass industrial production for which the problem of heat treatment and induction heating must be solved in the best possible way, i.e. with the optimum heating parameters with regard to the thermal process itself, or because of economic and technical aspects of the heating process, e.g. high heating rate, minimum electric energy consumption, required temperature distribution, etc.

It is common practice in industry to heat flat objects unilaterally or bilaterally by induction i.e. by means of an appropriately shaped heating inductor which is applied to one side of the heated object, or by means of two inductors applied to both sides of it. The inductors are the source of an electromagnetic wave which penetrates the surface of the heated object, supplying the electric energy converted into heat.

The electromagnetic field which has been generated in this way is characterized by the three vectors: the intensity of the magnetic field $\boldsymbol{H}$, of the electric field $\boldsymbol{E}$, and by the Poynting vector $\boldsymbol{S}$. The corresponding induction system with the inductor w and with the three vectors marked on the surface of a flat object p of a thickness d are shown schematically in Fig. 1.

In the case shown in Fig. 1a the object is heated unilaterally, while in the other two cases the object is heated bilaterally by means of two inductors; in the case 1b the induced currents are compatible with a regard to their direction and phase, e.g. both inductors w are supplied paralelly or in series from the same source; in case 1c the induced currents are inversed, e.g. both inductors are supplied separately from two different sources. All the three

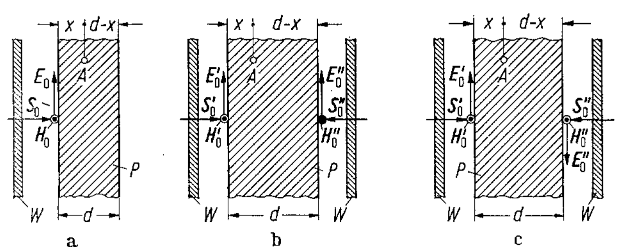

Fig. 1. Induction heating system of plates: a) unilateral heating, b) bilateral heating with compatible currents, c) bilateral heating with inverse currents

vectors of the generated field are perpendicular to each other in all the three induction systems shown in Fig. 1. We assume idealized operation conditions for these systems, i.e. that the inductors are fed with sinusoidal currents within a frequency range used in inductive heating, and the physical properties of the material of the object, i.e. its permeability $\mu = \mu_0 \mu_r$ and conductivity γ are constant and independent of the heating temperature and of the intensity of the magnetic field.

It is assumed further on that both the heated plate and the inductors extend infinitely along and across, and for our purposes we shall limit our considerations to tome section of such an inductive system. In this case we shall have to do with a homogeneous plane wave falling on one or on both sides of the metal plate, the vectors $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{S}_0$ on its surface being identical at each point. The changes of the field occur only inside the plate, in its thickness d. Thus the field parameters, e.g. at an arbitrary point A at a distance x from the surface will be exclusively the function of the distance x.

Of the three induction systems shown in Fig. 1 the system 1a illustrating a unilateral heating of the plate may be assumed as the basic one, since the systems 1b and 1c may be regarded as a superposition of two systems of the type 1a on both sides of the plate.

Though in the literature [1—3] cases of propagation of an electromagnetic plane wave have been discussed theoretically they have been treated only partially. If we discuss this problem on a broader basis, however, as a problem of induction heating of flat objects we may reach conclusions and results, which are of interest for practical application. Such an approach to this problem is the subject of the present paper.

2. Unilateral heating of a plate

The characteristics of a heated flat metal plate of any thickness may be known by determining the distribution of the electromagnetic field inside it, i.e. the intensity of the electric and the magnetic fields, the current density and the power density W/cm³ in the plate. For this purpose it is best to start with Helmholtz general differential equation which determines the field intensity H according to the distance x, namely:

$$\frac{\partial^2 \boldsymbol{H}}{\partial x^2} - k^2 = 0\,, \tag{1}$$

where at the depth penetration:

$$g = \sqrt{\frac{2}{\omega \gamma \mu_0 \mu_r}}$$

there is the argument:

$$k = \frac{1 + \mathrm{j}}{g} = \sqrt{2j}\,\sqrt{\frac{\omega \mu_0 \mu_r \gamma}{2}}\,.$$

A general solution of Eq. (1) is the binomial of the exponential functions:

$$H = A \exp(-kx) - B \exp(kx) \tag{2}$$

whereas the density of the electric field is determined on the basis of Maxwell's law:

$$\boldsymbol{E} = \frac{1}{\gamma} \operatorname{rot} \boldsymbol{H}$$

by differentiating Eq. (2) and considering that in the system of the orthogonal coordinates we have:

$$\operatorname{rot} \boldsymbol{H} = -\frac{\partial \boldsymbol{H}}{\partial x}\,.$$

We obtain then a formula analogical to (2), determining the intensity of the electric field:

$$E = \frac{k}{\gamma}\left[A \exp(-kx) - B \exp(kx)\right]. \tag{3}$$

Both formulae: (2) and (3) determine the distribution of the electromagnetic field inside the plate depending on its thickness x. On the surface of the plate where the wave is entering, we have the assumed field intensity H_0 and E_0, on the other, outer surface of the plate we shall already have different intensity values: let's denote them by H_1 and E_1. This intensity can be also determined on the basis of Maxwell's laws and on the assumption that the space behind the plate extends infinitely and does not conduct the current, having a dielectric constant $\varepsilon = \varepsilon_0 \cdot \varepsilon_r$, namely:

$$\operatorname{rot} \boldsymbol{H}_1 = \varepsilon_0 \varepsilon_r \frac{\partial \boldsymbol{E}_1}{\partial t}\,, \qquad \operatorname{rot} \boldsymbol{E}_1 = -\mu_0 \mu_r \frac{\partial \boldsymbol{H}_1}{\partial t}\,. \tag{4}$$

Assuming, as before, that both H_1 and E_1 are sinusoidal, the first from the above equations (4) will take the following form:

$$\operatorname{rot} \boldsymbol{H}_1 = \mathrm{j}\omega \varepsilon_0 \varepsilon_r \boldsymbol{E}_1 \tag{5}$$

After appropriate transformations of the second equation (4) we obtain successively:

$$\frac{1}{\mathrm{j}\omega \varepsilon_0 \varepsilon_r} \operatorname{rot}\operatorname{rot} \boldsymbol{H}_1 = -\mu_0 \mu_r \frac{\partial \boldsymbol{H}_1}{\partial t} = -\mathrm{j}\mu_0 \mu_r \omega \boldsymbol{H}_1\,,$$

$$\frac{\partial^2 H_1}{\partial x^2} = \mathrm{j}^2 \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \mu_r H_1\,.$$

Assuming in the considered space behind the plate $\varepsilon_r = 1$ and $\mu_r = 1$, and putting

$$k_1^2 = j^2\omega^2\varepsilon_0\mu_0$$

we obtain the final differential equation

$$\frac{\partial^2 H_1}{\partial x^2} - k_1^2 H_1 = 0 .$$

A general solution of this equation is the relation:

$$H_1 = C \exp(-k_1 x) - D \exp(k_1 x) . \qquad (6)$$

The second member of the above equation (6) represents a reflected wave, and considering that such a wave doesn't exist as the space behind the plate extends into infinity, there must be $D = 0$. Hence the last equation (5) assumes the form

$$H_1 = C \exp(-k_1 x) . \qquad (7)$$

The intensity of the electric field behind the plate may be determined in the basis of Eq. (5):

$$E_1 = \frac{1}{j\omega\varepsilon_0} \operatorname{rot} H_1 = \frac{k_1}{j\omega\varepsilon_0} C \exp(-k_1 x) . \qquad (8)$$

In the obtained relations (2), (3) and (7), (8) which determine the intensity of the field inside the plate and behind it, there appear the integration constants A, B, C which can be determined from the boundary conditions of the considered inductive system, namely:

$$\text{for } x = 0 \qquad H = H_0$$

$$\text{for } x = d \qquad H_1 = H_{1d} \quad \text{and} \quad E_1 = E_{1d}$$

Putting these conditions into the general relations (2) (3) and (7) (8) we obtain a set of three equations which allow to determine the looked for integration constants:

$$H_0 = A + B$$

$$A \exp(-kd) + B \exp(kd) = C \exp(-k_1 d) \qquad (9)$$

$$\frac{k}{\gamma} [A \exp(-kd) - B \exp(kd)] = \frac{k_1}{j\omega\varepsilon_0} C \exp(-k_1 d)$$

Dividing the last two equations the one by another we obtain the following relation:

$$\frac{A \exp(-kd) + B \exp(kd)}{A \exp(-kd) - B \exp(kd)} = \frac{j\omega k \varepsilon_0}{\gamma k_1} \qquad (10)$$

in which the right hand side practically approximates zero.

In induction heating, within the range of the applied frequencies, we have to do with various metals in various heating states. Thus we have, e.g.:

for copper at temp. 20 °C and 50 Hz

$$\varrho = 1.75 \cdot 10^{-6}\ \Omega\ \text{cm}; \ g \approx 0.9\ \text{cm}$$

for molten copper at 50 Hz

$$\varrho \approx 25.10^{-6}\ \Omega\ \text{cm}; \ g \approx 4\ \text{cm}$$

for molten cast-iron at 50 Hz

$$\varrho \approx 140.10^{-6}\ \Omega\ \text{cm}; \ g \approx 9\ \text{cm}$$

for wrought steel at temp. 20 °C and at 1 MHz

$$\varrho \approx 4.10^{-6}\ \Omega\ \text{cm}; \ g \approx 0.003\ \text{cm} .$$

Thus the value of relation (10) within the range of the practically used metals is

$$\frac{j\omega k\varepsilon_0}{\gamma k_1} = (1 + j) \frac{\varrho}{g} \sqrt{\frac{\varepsilon_0}{\mu_0}} = (1 + j)(5.2 \cdots 3.550) \cdot 10^{-10} \approx 0 .$$

Consequently we may write

$$A \exp(-kd) = - B \exp(kd)$$

$$B = - A \exp(-2kd)$$

By means of the first from the group of the three equations (9) we may finally determine the constants A and B, namely

$$H_0 = A [1 - \exp(-2kd)] ,$$

$$A = H_0 \frac{1}{1 - \exp(-2kd)} , \qquad (11)$$

$$B = - H_0 \frac{\exp(-2kd)}{1 - \exp(-2kd)} ,$$

From the second equation of that group (9) it follows that

$$H_1 = C \exp(-k_1 d) = 0 .$$

Hence the intensity of the magnetic field behind the metal plate equals zero. Thus, for the range of industrial frequencies and kinds of metals practically used in induction heating the electromagnetic wave does not penetrate the plate which behaves similarly to a magnetic screen.

Basing on the calculated integration constants (11) it is possible to determine the field parameters inside the plate, namely

$$H = H_0 \frac{\exp(-kx) - \exp(-2kd) \exp(kx)}{1 - \exp(-2kd)} = H_0 \frac{\exp[k(d - x)] - \exp[-k(d - x)]}{\exp(kd) - \exp(-kd)}$$

and replacing the exponential functions by the hyperbolic ones we obtain the final relation for the intensity of the magnetic field inside the plate

$$H = H_0 \frac{\sinh [k(d-x)]}{\sinh (kd)} \tag{12}$$

whereas the intensity of the electric field may be determined by means of the first Maxwell equation

$$E = H_0 \frac{k}{\gamma} \frac{\cosh [k(d-x)]}{\sinh (kd)} \tag{13}$$

as well as the current density inside the plate:

$$i_0 = H_0 k \frac{\cosh [k(d-x)]}{\sinh (kd)} . \tag{14}$$

The fourth quantity characterizing the energetic properties of the field is the volume density of the heating power W/cm³ inside the plate, which can be determined from Joule's law, using Eq. (14):

$$p = \frac{1}{2\gamma} i^2 = \frac{1}{2\gamma} (H_0 k)^2 \frac{\cosh^2 [k(d-x)]}{\sinh^2 (kd)} . \tag{15}$$

The above determined four parameters characterize sufficiently the field distribution inside the plate, but the most convenient way to find out its properties is to represent it in the form of a graph. For this purpose the above field parameters must be defined as absolute quantities, and then we obtain successively

$$|H| = H_0 \sqrt{\frac{\cosh \frac{2(d-x)}{g} - \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} - \cos \frac{2d}{g}}} , \tag{16}$$

$$|E| = \frac{\sqrt{2}}{\gamma g} H_0 \sqrt{\frac{\cosh \frac{2(d-x)}{g} + \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} - \cos \frac{2d}{g}}} , \tag{17}$$

$$|i| = \frac{2}{g} H_0 \sqrt{\frac{\cosh \frac{2(d-x)}{g} + \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} - \cos \frac{2d}{g}}} , \tag{18}$$

$$|p| = \frac{1}{\gamma} \left(\frac{H_0}{g}\right)^2 \frac{\cosh \frac{2(d-x)}{g} + \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} - \cos \frac{2d}{g}} . \tag{19}$$

To represent graphically the propagation of the electromagnetic wave in an inductively heated plate the best way is to consider the ratio of the field intensity, current and power density inside the plate to the field intensity, current and power density occurring on its surface, i.e. H/H_0 and E/E_0, i/i_0 and p/p_0. Determining thus the field parameters for $x = 0$ by means of relations (16) to (19) we obtain

$$\frac{|H|}{|H_0|} = \sqrt{\frac{\cosh \frac{2(d-x)}{g} - \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} - \cos \frac{2d}{g}}} , \tag{20}$$

$$\frac{|E|}{|E_0|} = \frac{|i|}{|i_0|} = \frac{\cosh \frac{2(d-x)}{g} + \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} + \cos \frac{2d}{g}} , \tag{21}$$

$$\frac{|p|}{|p_0|} = \frac{\cosh \frac{2(d-x)}{g} + \cos \frac{2(d-x)}{g}}{\cosh \frac{2d}{g} + \cos \frac{2d}{g}} . \tag{22}$$

We shall represent graphically the particular values of the above relations (20), (21), (22) depending on x/d for plates of different thickness, determined by the ratio d/g as well as for various metals and frequencies used practically in industry. They have been calculated for the whole range $x/d = 0 \cdots 1.0$ and for the arguments corresponding to practical applications in industry, thus for $d/g = 0.5 \cdots 10$ to 50. They have been put together in Table 1.

On the basis of the data from Table 1 there have been prepared diagrams shown in Fig. 2.

The diagrams show the propagation of an electromagnetic field inside the plates. Their character is in general known from the literature, however the practical conclusions which can be drawn from them may be of interest for induction heating of flat objects.

The greatest field parameters occur on the surface of a plate where the wave is falling, and inside the plate they become smaller reaching minimum values on the reverse surface of the plate. Thus the field distribution inside the plate is not uniform, the greatest intensity of the magnetic field H_0 being on the surface, and on the other side $H = 0$. With the small ratios $d/g \leqq 1$ the intensity decreases nearly linearly, with greater ratios the distribution of H resembles an exponential function.

The current density i and at the same time the intensity of the electrical field E in the plate are not distributed uniformly, either. When the plates are not very thick, with $d/g \leqq 1$, the difference in uniformity is not too great, and then the current density on the opposite surface of the plate diminishes by ca. 23% of the current density on the entrance surface. However, when the plate is somewhat thicker,

Table 1. Electromagnetic field parameters in a plate heated unilaterally

	$\frac{x}{d}$	$\frac{d}{g}$ 0.5	1.0	2.0	3.0	6.0	10.0	50
$\frac{H}{H_0}$	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	0.25	0.75	0.73	0.63	0.48	0.22	0.08	0.0
	0.5	0.50	0.49	0.39	0.23	0.05	0.0	0.0
	0.75	0.25	0.24	0.19	0.11	0.01	0.0	0.0
	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\frac{E}{E_0} = \frac{i}{i_0}$	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	0.25	0.99	0.85	0.58	0.47	0.22	0.08	0.0
	0.5	0.98	0.79	0.35	0.25	0.05	0.0	0.0
	0.75	0.98	0.78	0.28	0.10	0.01	0.0	0.0
	1.0	0.98	0.77	0.27	0.09	0.01	0.0	0.0
$\frac{p}{p_0}$	0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	0.25	0.98	0.72	0.34	0.22	0.05	0.01	0.0
	0.5	0.96	0.62	0.13	0.05	0.0	0.0	0.0
	0.75	0.96	0.60	0.08	0.01	0.0	0.0	0.0
	1.0	0.96	0.60	0.08	0.01	0.0	0.0	0.0

e.g. $d/g = 2$, then the loss of current density amounts to ca. 73%.

The distribution of power density inside a plate shown in Fig. 2c is of greater interest for induction heating.

When the plates are not thick, with $d/g \leqq 1$, induction heating is fairly uniform. With $d/g = 1$, the loss of power density equals 40%. This condition will be of interest for a thorough induction heating of metal plates. With plates of greater thickness however, the generated Joule's heat is distributed very unevenly, and thus, for example, with $d/g = 2$ the power density on the reverse side of the plate decreases by 92% of the power density on the entrance surface. With still thicker plates, when $d/g > 3$ the production of heat diminishes entirely already at $x/d \approx 0.5$.

From the practical point of view we are mostly interested in the total heating power formed inside the plate which may be determined by means of relation (15) and (19). We shall determine it for a unit surface 1 cm² of the plate cross section in W/cm².

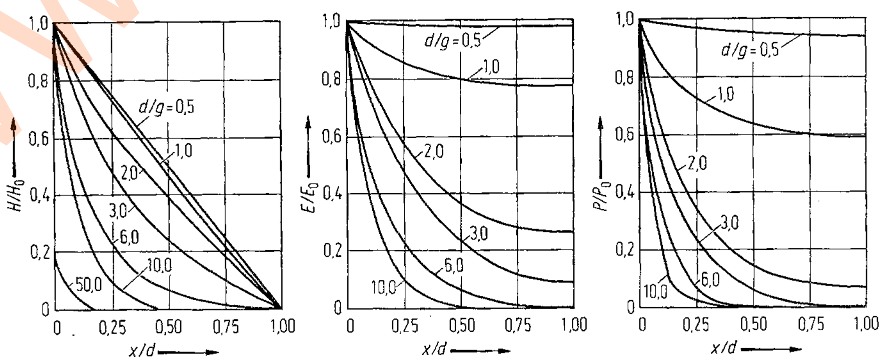

Fig. 2. Distribution of field parameters inside a plate heated unilaterally: a) magnetic field intensity, b) electrical field intensity, c) power density

$$P = \int_0^d p\,\mathrm{d}x = \frac{1}{2\gamma}(H_0 k)^2 \int_0^d \frac{\cosh^2[k(d-x)]}{\sinh^2(kd)}\,\mathrm{d}x =$$

$$= \frac{1}{\gamma}\left(\frac{H_0}{g}\right)^2 \int_0^d \frac{\cosh\frac{2(d-x)}{g} + \cos\frac{2(d-x)}{g}}{\cosh\frac{2d}{g} - \cos\frac{2d}{g}}\,\mathrm{d}x =$$

$$= \frac{H_0^2}{2g\gamma} \cdot \frac{\sinh\frac{2d}{g} + \sin\frac{2d}{g}}{\cosh\frac{2d}{g} - \cos\frac{2d}{g}}\,. \qquad (23)$$

It can be seen from the above relation (23) that the total power of the plate depends not only on the electrical parameters of the plate and the field intensity $\boldsymbol{H}$, but also on the thickness of the plate, or rather on the ratio d/g, which is taken into consideration by the coefficient

$$\lambda = \frac{\sinh\frac{2d}{g} + \sin\frac{2d}{g}}{\cosh\frac{2d}{g} - \cos\frac{2d}{g}}\,.$$

This coefficient is the function of the ratio d/g and when the plate thickness d increases infinitely relative to the penetration depth g, it assumes the value $\lambda = 1$, and then the heating power in the plate is defined by the formula

$$P = \frac{H_0^2}{2g\gamma}\,. \qquad (24)$$

From the point of view of the heating power produced in the plate we are interested in the change of the function $\lambda = f(d/g)$, which is represented by the diagram in Fig. 3 made on the basis of Table 2.

From the diagram shown in Fig. 3 it follows that the greatest heating powers are formed inside thin plates, when their thickness is $d/g < 1,2$. Starting with the ratio $d/g \geqq 1.2$ the value of the function is $\lambda = 1$, with only small deviations, and then the heating power also retains approximately the value defined by relation (24). With $d/g = \pi/2$ function λ reaches its minimum, and then the heating power also reaches its minimum value, equal to 92% of the power defined by relation (24). With the plate thickness increasing further on, the heating power remains approximately the same, its value being defined by formula (24). Then the function λ attains its minimum and maximum values, which, however, are close to 1, which from the practical point of view can be neglected. Thus, for example, with $d/g = \pi$ we have $\lambda = 1.004$.

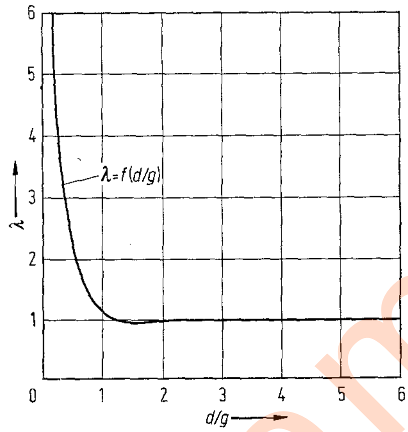

Fig. 3. Diagramm of the function $\lambda = f(d/g)$ of a plate heated unilaterally

The thickness of a plate may theoretically increase up to $d = \infty$, and then we have an extreme case of unilateral heating of a very thick plate. The distribution of the electromagnetic field in such plates can be determined by means of simple mathematical formulae. Starting with formulae (12) and putting in it $d = \infty$ we obtain after mathematical transformations the familiar relation which determines the magnetic field intensity in a thick plate:

$$H = H_0 \exp(-kx) \qquad (25)$$

and having taken into consideration the absolute quantities we have

$$|H| = H_0 \exp\left(-\frac{x}{g}\right). \qquad (26)$$

Using the same method it is possible to determine the intensity of the electrical field and the current density starting with the formulae (13) and (14):

$$E = \frac{k}{\gamma} H_0 \exp(-kx)\,, \qquad i = kH_0 \exp(-kx) \qquad (27)$$

Table 2. Function $\lambda = f\left(\frac{d}{g}\right)$ of a plate heated unilaterally

$\frac{d}{g}$	0	0.1	0.2	0.3	0.5	0.75	1.0	1.57	2.0	3.14	5.0	10.0
λ	∞	10.00	5.00	3.34	2.01	1.38	1.09	0.92	0.95	1.00	1.00	1.00

and in terms of absolute quantities we have:

$$|E| = \frac{\sqrt{2}}{g\gamma} H_0 \exp\left(-\frac{x}{g}\right),$$
$$|i| = \frac{\sqrt{2}}{g} H_0 \exp\left(-\frac{x}{g}\right). \tag{28}$$

Whereas the power density calculated by means of Joule's law and current density (28) is

$$|p| = \frac{1}{2}\gamma E^2 = \frac{1}{\gamma}\left(\frac{H_0}{g}\right)^2 \exp\left(-\frac{2x}{g}\right) \tag{29}$$

while the total heating power produced in a plate of a thickness $d = \infty$, in its cross section 1 cm² is

$$P = \int_0^\infty p\,\mathrm{d}x = \frac{H_0^2}{2g\gamma}. \tag{30}$$

The propagation of an electromagnetic field of a thick plate heated unilaterally by induction can be obtained by relating the changes of its parameters to the parameters occurring on the entrance surface of the plate, i.e.

$$\frac{H}{H_0} = \frac{E}{E_0} = \frac{i}{i_0} = \mathrm{e}^{-x/g} \quad \text{and} \quad \frac{p}{p_0} = \mathrm{e}^{-2x/g}$$

depending on x/g.

This has been shown below, in Fig. 4 basing on the values of function $\mathrm{e}^{-x/g}$ and $\mathrm{e}^{-2x/g}$, included in Table 3.

Table 3. The changes of the values $\mathrm{e}^{-x/g}$ and $\mathrm{e}^{-2x/g}$

x/g	0.0	0.5	1.0	2.0	3.0	5.0
$\mathrm{e}^{-x/g}$	1.0	0.61	0.37	0.14	0.05	0.01
$\mathrm{e}^{-2x/g}$	1.0	0.37	0.14	0.02	0.0	0.0

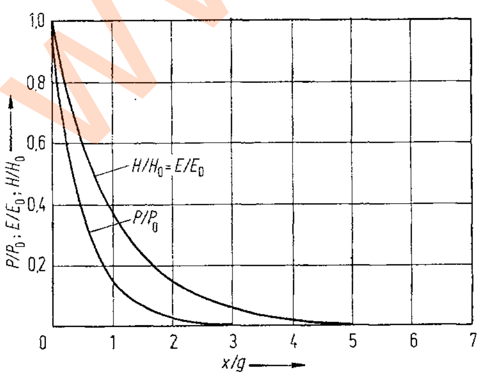

Fig. 4. Field parameters in a thick plate

When interpreting the results shown in Fig. 4 we can state that the field density and the current density practically disappear already at a depth $x = 5g$, whereas the power density p disappears at a depth $x = 3g$. At the depth $x = 3g$ the current density decreases to 5% of current density occurring on the surface, and the power density diminishes to $p = 0$. On that basis we can reach the practical conclusion that a flat metal plate of a thickness $d = 3g$, heated inductively by a plane electromagnetic wave may be regarded as a thick plate in which the wave propagation may be analyzed according to formulae (27) ··· (30). Flat, thinner plates of a thickness $d < 3g$ should be dealt with according to more complicated mathematical formulae (10) to (22).

3. Bilateral heating of a plate

A flat plate may be heated inductively bilaterally as illustrated by two examples shown in Figs. 1b and 1c, i.e. by means of currents induced on both sides of a plate and directed compatibly, or inversed. Let us first consider a case in which an arbitrary flat plate is heated bilaterally as shown in Fig. 1b. Then, according to the assumptions the intensities of the electric field E_0' and E_0'' occurring on both sides of the plate are equal regarding their moduli, phase and direction. However, their magnetic field densities H_0' and H_0'' have opposite directions and their moduli H_0 are equal as well.

Inside the plate the induced currents will flow in one and the same direction corresponding to the intensity of the electric field. At an arbitrary point A, at a distance x from the left surface of the plate there will appear the electric field intensities E' and E'' and the magnetic field intensities H' and H'', the resultant field parameters being

$$H = H' - H'',$$
$$E = E' + E''.$$

Both parameters given above may be determined on the basis of the already derived relations (12) and (13):

$$H' = H_0 \frac{\sinh[k(d-x)]}{\sinh(kd)},$$

$$H'' = H_0 \frac{\sinh(kx)}{\sinh(kd)}.$$

Hence the magnetic field intensity at a point A is

$$H = H_0 \frac{\sinh[k(d-x)] - \sinh(kx)}{\sinh(kd)}$$

and after mathematical transformations we obtain finally:

$$H = H_0 \frac{\sinh\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh\left(k\frac{d}{2}\right)}. \tag{31}$$

In the same way we can calculate the intensity of the electric field at a point A:

$$E = H_0 \frac{k}{\gamma} \frac{\cosh\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh\left(k\frac{d}{2}\right)} \tag{32}$$

and the current density

$$i = H_0 k \frac{\cosh\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh\left(k\frac{d}{2}\right)}. \tag{33}$$

The energy parameter of the field, i.e. the density of the heating power inside the plate is found from Joule's law:

$$p = \frac{1}{2\gamma} i^2 = \frac{1}{2\gamma} (H_0 k)^2 \frac{\cosh^2\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh^2\left(k\frac{d}{2}\right)}. \tag{34}$$

The above relations (31) to (34) determine sufficiently the propagation of an electromagnetic wave in a metal plate heated bilaterally by induction. However, the distribution of this field in a plate can be best represented similarly as before, i. e. by means of a graph. For this purpose the most convenient method is to relate the field parameters in the plate to the parameters on its surfaces, i.e. H/H_0, E/E_0, i/i_0, and p/p_0.

Calculating thus the field parameters for $x = 0$ and starting from formulae (31) to (34) we obtain

$$\frac{H}{H_0} = \frac{\sinh\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh\left(k\frac{d}{2}\right)},$$

$$\frac{E}{E_0} = \frac{i}{i_0} = \frac{\cosh\left[k\left(\frac{d}{2} - x\right)\right]}{\cosh\left(k\frac{d}{2}\right)},$$

$$\frac{p}{p_0} = \frac{\cosh^2\left[k\left(\frac{d}{2} - x\right)\right]}{\cosh^2\left(k\frac{d}{2}\right)}.$$

The values of these relations can be represented graphically at different depths of the plate, depending on x/d, and for different thickness of the plate determined by the ratio $d/g = 0.5 \cdots 50$, for different metals and frequencies used in industry. For this purpose we must replace the complex functions by functions expressed in absolute quantities, and then we have:

$$\frac{|H|}{|H_0|} = \sqrt{\frac{\cosh\frac{d-2x}{g} - \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} - \cos\frac{d}{g}}}, \tag{35}$$

$$\frac{|E|}{|E_0|} = \frac{|i|}{|i_0|} = \sqrt{\frac{\cosh\frac{d-2x}{g} + \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}}}, \tag{36}$$

$$\frac{|p|}{|p_0|} = \frac{\cosh\frac{d-2x}{g} + \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}}. \tag{37}$$

On the basis of detailed numerical values calculated for the particular relations (35), (36) and (37) the field parameters of a plate heated bilaterally are shown in Fig. 5.

As it is found, the parameters in each half of the plate are identical with those of a plate heated unilaterally, which are shown in Fig. 3.

The greatest field parameters occur on both outer surfaces of the plate, and inside it they decrease to a minimum. Thus, e.g. inside the plate $H = 0$, and the electric field E and power density p show the smallest values. The irregularity of the field distribution is thus as large as in case of a plate heated unilaterally, and all the conclusions referring to the wave propagation, given before, can be applied here, as well.

The total heating power formed inside a plate heated bilaterally is

$$P = \int_0^d p \, dx = \frac{1}{2\gamma} (H_0 k)^2 \int_0^d \frac{\cosh^2\left[k\left(\frac{d}{2} - x\right)\right]}{\sinh^2\left(k\frac{d}{2}\right)} dx =$$

$$= \frac{1}{\gamma}\left(\frac{H_0}{g}\right)^2 \int_0^d \frac{\cosh\frac{d-2x}{g} + \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} - \cos\frac{d}{g}} dx =$$

$$= \frac{H_0^2}{2g\gamma} \cdot \frac{\sinh\frac{d}{g} + \sin\frac{d}{g}}{\cosh\frac{d}{g} - \cos\frac{d}{g}}. \tag{38}$$

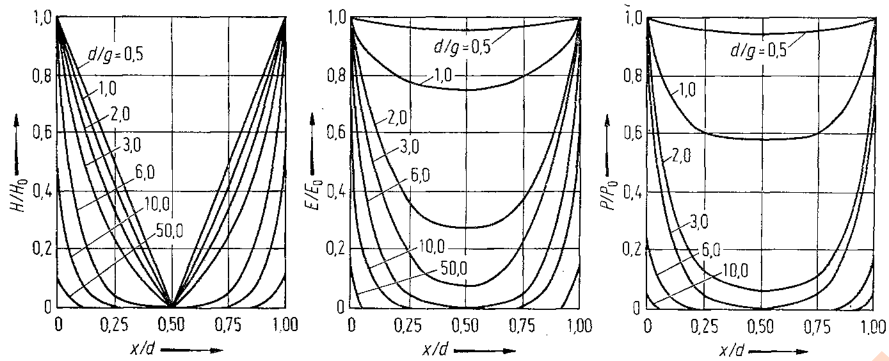

Fig. 5. Field parameters in a plate heated bilaterally with compatible currents

Similarly as before, this total heating power depends either on the thickness of the plate or on the ratio d/g, this relation being expressed by the coefficient

$$\lambda = \frac{\sinh \frac{d}{g} + \sin \frac{d}{g}}{\cosh \frac{d}{g} - \cos \frac{d}{g}} . \tag{39}$$

This is the same relation that has already been indicated by relation for a unilateral heating of a plate, except that in formula (39) the independent variable d is twice as large. On the basis of the graph given before in Fig. 3 we may reach the following practical conclusions concerning induction heating:

The greatest heating power in plates heated bilaterally with compatible currents is formed when their thickness is $d < 2.4g$. In thicker plates, when $d \geqq 2.4g$ the value of function λ according to formula (39) is $\lambda \approx 1$, and the total heating power retains, with only small deviations, the same value as for a plate heated unilaterally, which is determined by formula (24), independent of the plate thickness.

With the thickness $d = \pi \cdot g$ the total heating power reaches its minimum, which amounts to 92% of the maximum power, according to relation (24).

What concerns the results of analysis of the field parameters in a thick plate heated unilaterally and shown in Fig. 4, we may also in this case maintain that a metal plate of a thickness $d = 6g$, heated bilaterally, may be practically regarded as a thick plate whose field propagation may be analysed according to relations (27) to (30). The calculations for thinner plates of a thickness $d < 6g$ should be carried out according to mathematical formulae (35) to (38).

Let us consider now a case of a flat plate heated bilaterally by inverse currents of equal moduli which is illustrated by the induction system shown in Fig. 1c. With the assumed directions of the Poynting vectors S_0' and S_0'' of equal values, falling on both sides of the plate, and with inversed and equal vectors E_0' and E_0'' the vectors of the magnetic field intensity have the same direction and their moduli are equal.

At an arbitrary point A of a plate there appear the resultant field intensities, i.e.

$$H = H' + H''$$

$$E = E' - E'' .$$

Making use of the already obtained relations (12) and (13) which determine the field intensity components, and after mathematical transformations, we obtain:

$$H = H_0 \frac{\cosh \left[k \left(\frac{d}{2} - x \right) \right]}{\cosh k \frac{d}{2}} , \tag{40}$$

$$E = \frac{k}{\gamma} H_0 \frac{\sinh \left[k \left(\frac{d}{2} - x \right) \right]}{\cosh k \frac{d}{2}} \tag{41}$$

the current and the power densities being as follows:

$$i = H_0 k \cdot \frac{\sinh \left[k \left(\frac{d}{2} - x \right) \right]}{\cosh k \frac{d}{2}} , \tag{42}$$

$$p = \frac{1}{2\gamma} (H_0 k)^2 \frac{\sinh^2 \left[k \left(\frac{d}{2} - x \right) \right]}{\cosh^2 k \frac{d}{2}} . \tag{43}$$

Calculating in the same way as before the field parameters of a plate in absolute quantities and relating them to parameters occurring on the sur-

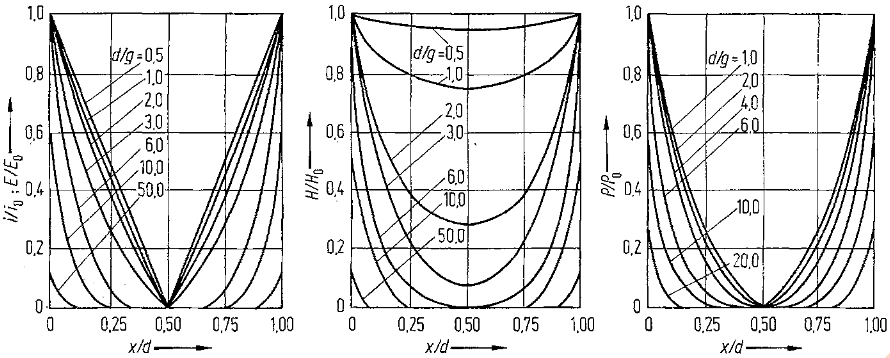

Fig. 6. Field parameters in a plate heated bilaterally by inverse currents

face of a plate, we have:

$$\frac{|H|}{|H_0|} = \sqrt{\frac{\cosh\frac{d-2x}{g} + \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}}}, \tag{44}$$

$$\frac{|E|}{|E_0|} = \frac{|i|}{|i_0|} = \sqrt{\frac{\cosh\frac{d-2x}{g} - \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} - \cos\frac{d}{g}}}, \tag{45}$$

$$\frac{|p|}{|p_0|} = \frac{\cosh\frac{d-2x}{g} - \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} - \cos\frac{d}{g}}. \tag{46}$$

When comparing the first two of the above relations (44) and (45) with relations (35) and (36) we can see that the intensity distribution of the magnetic field in a plate heated bilaterally by means of inverse currents is identical with the intensity distribution of the electrical field in a plate heated by compatible currents. The same refers to the electrical field intensity. On the basis of this statement it is easy to represent both these cases graphically, as shown in Fig. 6.

Table 4. Power density in a plate by inverse currents

	x/d	d/g					
		1.0	2.0	4.0	6.0	10.0	20.0
p/p_0	0.0	1.0	1.0	1.0	1.0	1.0	1.0
	0.10	0.64	0.62	0.48	0.30	0.14	0.02
	0.25	0.25	0.24	0.15	0.06	0.01	0.0
	0.5	0.0	0.0	0.0	0.0	0.0	0.0
	0.75	0.25	0.24	0.15	0.06	0.01	0.0
	0.90	0.64	0.62	0.48	0.30	0.14	0.02
	1.0	1.0	1.0	1.0	1.0	1.0	1.0

The power density, however, runs differently in the plate, and in order to represent it graphically in Fig. 6 we have made up Table 4 containing its particular values.

It follows from the above diagrams that the field distribution in a plate heated bilaterally by inverse currents is more uneven than the field in a plate heated bilaterally as well, but by compatible currents. This refers in particular to the current and power densities in the plate. Thus, e. g. in thick plates both these densities disappear, i. e. both $i = 0$, as well as $p = 0$ if $d > 6g$, even in the fairly wide middle part of the plate. Considering this fact as well as taking into account the thorough heating of the plate it is the most disadvantageous method of induction heating of flat metal objects.

From the point of view of utility the most important quantity is the total heating power of the plate which is defined as follows:

$$P = \int_0^d p\,\mathrm{d}x = \frac{1}{\gamma}\left(\frac{H_0}{g}\right)^2 \int_0^d \frac{\cosh\frac{d-2x}{g} - \cos\frac{d-2x}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}}\,\mathrm{d}x =$$

$$= \frac{H_0^2}{2g\gamma} \cdot \frac{\sinh\frac{d}{g} - \sin\frac{d}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}}. \tag{47}$$

This power depends also on the plate thickness d, and this fact is indicated in formula (47) by the coefficient

$$\lambda = \frac{\sinh\frac{d}{g} - \sin\frac{d}{g}}{\cosh\frac{d}{g} + \cos\frac{d}{g}} \tag{48}$$

Table 5. Function λ of a plate heated bilaterally by inverse currents

d/g	0	0.5	1.0	2.0	π	5.0	2π	8	3π	10	4π
λ	0	0.02	0.16	0.82	1.09	1.01	0.99	1.00	1.00	1.00	1.00

which constitutes the function $\lambda = f(d/g)$ which changes its value within the range from $\lambda = 0$ to $\lambda = 1$, while the thickness of the plate changes within the limits $d = 0$ to $d = \infty$. This function is represented by a graph Fig. 7 drawn on the basis of Table 5.

This function assumes changing values of λ; at $d/g \leqq 2.4$ we have $\lambda \leqq 1$, whereas at $d/g > 2.4$ $\lambda \approx 1$, reaching its maximum $\lambda = 1.09$ at $d/g = \pi$. If the thickness of the plate increases to $d = \infty$, then the total heating power becomes stable, and practically no longer depends on the thickness of the plate. It is expressed by the following formula:

$$P = \frac{H_0^2}{2g\gamma}.$$

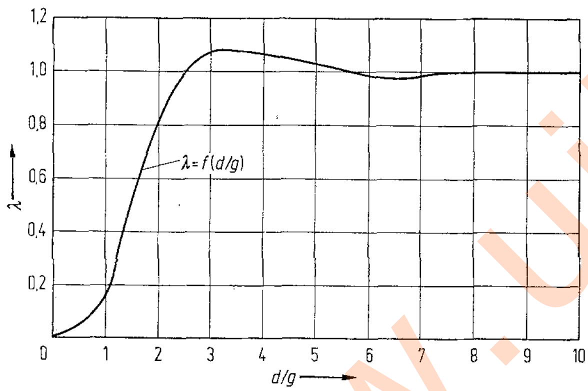

Fig. 7. Graph of the function $\lambda = f(d/g)$ of a plate heated bilaterally by inverse currents

Reaching practical conclusions from function λ and from relation (47) we can state that flat objects of a thickness $d \geqq 2{,}4\,g$ can be advantageously heated by inverse currents. With lesser thickness of plates the total heating power decreases rapidly since the inverse currents annihilate each other. At a thickness $d = \pi g$ the heating power reaches its maximum value which is only by 9% greater than the heating power of a very thick plate, for which $\lambda = 1$.

This kind of bilateral induction heating of flat plates by means of inverse currents could be recommended for surface heating, e. g. for the purpose of hardening thick plates, in which, on account of the annihilation process of inverse currents the heating of both layers shows greater "contrast" than this is the case in plates heated by compatible currents. When heating is being done by means of inverse currents both the heated "layers" of the plate become more distinguished by heat production from the rest, i. e. from the deeper layers of the plate.

Summary

In the above discussion we have taken into consideration flat objects only, heated in three different ways. Yet it is possible to prove that the reasoning, calculations and conclusions given above can be applied to plates of a cylindrical shape, the outer diameter D of which satisfies the following condition:

$$\frac{D}{g}\sqrt{2} \geqq 16 .$$

Then, admittedly, we shall have to deal with a cylindrical propagation of an electromagnetic wave, but it can be replaced by a plane wave at such great outer diameters of the cylinders. The calculation errors made in this case are practically negligible. Thus the calculations and conclusions given above attain greater importance for practical application in industry.

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